Strong ergodicity phenomena for Bernoulli shifts of bounded algebraic dimension

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Dynamics and orbit equivalence relations

Given a continuous action $G \curvearrowright X$ of a Polish group G on Polish space X we let E_X^G be the associated **orbit equivalence relation**:

$$xE_X^G x' \iff \exists g \in G \, (g \cdot x = x').$$

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(2) Its strong ergodic properties.

We say that (X, E) is **strongly ergodic** with respect to (Y, F) if for every Borel $f \colon X \to Y$ with $xEx' \implies f(x)Ff(x')$ there is a comeager $C \subseteq X$ so that for all $x, x' \in C$ we have that $f(x)Ff(x')$.

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Theorem (Solecki)

Let G be a Polish group. Then the following are equivalent:

1. G *is* **compact**;
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Question (Kechris)

Let G be a Polish group which is **not locally-compact***. Does there exist some action* $G \curvearrowright X$ *so that* (X, E^G_X) *is not* **essentially countable***?*

Polish permutation groups

Let $\mathrm{Sym}(\mathbb{N})$ be the Polish group of all bijections $g\colon \mathbb{N} \to \mathbb{N}$ endowed with the pointwise convergence topology.

A **Polish permutation group** P is any closed subgroup of $\mathrm{Sym}(\mathbb{N})$. Such P comes together with an action $P \curvearrowright \mathbb{N}$ with $(g, n) \mapsto g(n)$.

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Theorem (Kechris, Malicki, P., Zielinski)

If P is **not locally compact** *then $E_{\mathrm{inj}}(P)$ is not essentially countable. Similarly for when P is non-compact or non-CLI.*

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The **algebraic dimension** $\dim(P)$ of P is the smallest $n \in \mathbb{N}$ so that for all $A \subseteq \mathbb{N}$ with $|A| = n+1$, there is $a \in A$ so that $a \in [A \setminus \{a\}]_P$, if such n exists. Otherwise, we write $\dim(P) = \infty$.

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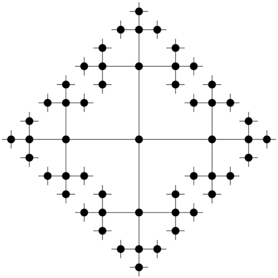
Examples.

(1) Let T_4 be the infinite 4-regular tree:



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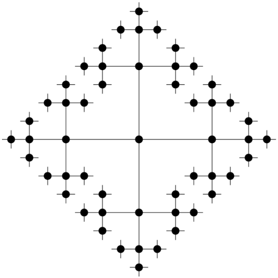

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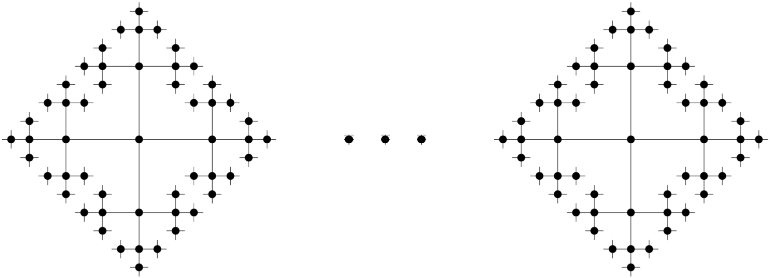

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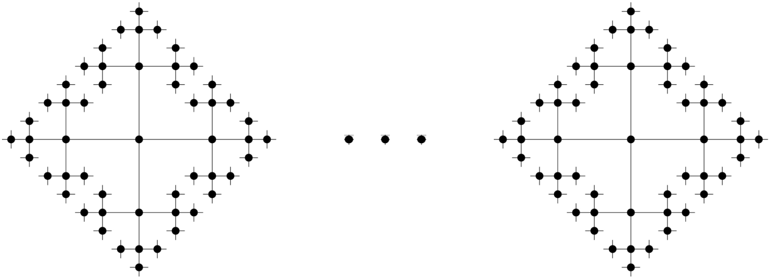

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This is a consequence of the fact that in all these examples the closure operator $A \mapsto [A]_P$ additionally satisfied the **exchange property**:

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There exist **non-locally compact** P with $\dim(P) < \infty$.

Bernoulli shifts and algebraic dimension

Let Q be a Polish permutation group. Recall the orbit equivalence relation:

$E_{\mathrm{inj}}(Q)$, induced on the injective part of the Bernoulli shift $Q \curvearrowright \mathrm{Inj}(\mathbb{N}, \mathbb{R})$.

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Let P and Q be Polish permutation groups and let $n \in \mathbb{N}$. Assume that:

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- P is **locally-finite** if for all finite $A \subseteq \mathbb{N}$ we have that $[A]_P$ is finite.
- P is $(n+1)$–**free** if for all finite $A \subseteq \mathbb{N}$ there are $g_0, g_1, \ldots, g_n \in P$ so that for all $i \leq n$ we have that $[g_i A]_P$ and $[\bigcup_{j:j\neq i} A_j]_P =$ are disjoint.

Some examples from Baldwin-Koerwien-Laskowski

$\mathcal{L}_{\mathbf{2}} = \{f_0, f_1, f_2, \ldots\}$ consists of a sequence of $\mathbf{2}$-ary function symbols.
Let $\mathbb{M}_{\mathbf{2}}$ be the Fraïssé limit of the class $\mathcal{K}_{\mathbf{2}}$ of all finite $\mathcal{L}$-structures $\mathbb{A}$ s.t.

1. for all a_0, a_1 in $\mathbb{A}$ and cofinitely many $n \in \mathbb{N}$, $f_n(a_0, a_1) = a_0$.
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Corollary. (P., Shani) If $m \lneq n$, then $E(P_{\boldsymbol{n}}))$ is strongly ergodic w.r.t. $E_{\mathrm{inj}}(P_{\boldsymbol{m}}))$. In particular, we have that:

$$E_{\mathrm{inj}}(P_{\mathbf{2}})) \lneq_B E_{\mathrm{inj}}(P_{\mathbf{3}})) \lneq_B E_{\mathrm{inj}}(P_{\mathbf{4}})) \lneq_B \cdots$$

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Example. Let $E_{\text{inj}}(\mathrm{Sym}(\mathbb{N}))$ be the injective part of $\mathrm{Sym}(\mathbb{N}) \curvearrowright \mathbb{R}^{\mathbb{N}}$:
$(x_n : n \in \mathbb{N}) E_{\text{inj}}(\mathrm{Sym}(\mathbb{N}))(y_n : n \in \mathbb{N}) \iff \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}$

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Question. What about the converse? Is there a nice basis for the class of unpinned equivalence relations under Borel reductions?

Table of Contents

Main theorem

Theorem (P., Shani)

Let P and Q be Polish permutation groups and let $n \in \mathbb{N}$. Assume that:

1. $\dim(Q) \leq n$;
2. P *is locally-finite and* $(n+1)$*–free.*

Then, $E(P)$ *is strongly ergodic against* $E_{\mathrm{inj}}(Q)$*. Hence,* $E(P) \not\leq_B E_{\mathrm{inj}}(Q)$

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The proof employs/builds on **symmetric model techniques**.

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Lemma. (*Existence of supports*) For any $S \in V(\{x_n^G\})$ with $S \subseteq V$ there is a finite $F \subseteq \{x_n^G \colon n \in \mathbb{N}\}$ so that $S \in V[F]$.

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To conclude:

Theorem (Shani)

Suppose E and F are Borel equivalence relations on X and Y respectively and $x \mapsto \mathcal{N}^x$ and $y \mapsto \mathcal{M}^y$ be classifications by countable structures of E and F respectively. Then, the following are equivalent.

1. *For every Borel homomorphism $f\colon (X_0, E) \to (Y, F)$, where $X_0 \subseteq X$ is non-meager, f maps a non-meager set into a single F-class;*
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In the case of the Bernoulli shifts, we have that $P = \mathrm{Aut}(\mathcal{N})$ and $Q = \mathrm{Aut}(\mathcal{M})$ for countable structures $\mathcal{M}$ and $\mathcal{N}$. So we have that:

$P \curvearrowright \mathrm{Inj}(\mathbb{N}, \mathbb{R})$ is classified by $(x_n : n \in \mathbb{N}) \mapsto$ "$\mathcal{N}$ on $\{x_n : n \in \mathbb{N}\}$"

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Thαnk you!